CHEBYSHEV APPROXIMATION FOR THE ROOTS OF THE EQUATION $\mathrm{Bi}W_{\Gamma}(\mu) = \mu V_{\Gamma}(\mu)$

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A minimax approximation, uniform for $\mathrm{Bi} \in [0, \infty)$, is developed for the roots of the equation $\mathrm{Bi}W_{\Gamma}(\mu) = \mu V_{\Gamma}(\mu)$, by means of Chebyshev polynomials.

Analytic determination of the temperature distribution in a one-dimensional body under all possible initial, internal, and external conditions requires finding the roots of the characteristic equation [1, 2]

$$\mathrm{Bi}\, W_{\Gamma}(\mu) = \mu V_{\Gamma}(\mu),\quad 0 \leqslant \mathrm{Bi} < \infty, \tag{1}$$

where the function $W_{\Gamma}(\mu)$ and $V_{\Gamma}(\mu)$ for a plate ($\Gamma = 0$), cylinder ($\Gamma = 1$), and sphere ($\Gamma = 2$) have the form

$$\begin{gathered} W_0(\mu) = \cos\mu,\; W_1(\mu) = J_0(\mu),\;\; W_2(\mu) = \frac{\sin\mu}{\mu}; \\ V_0(\mu) = \sin\mu,\; V_1(\mu) = J_1(\mu),\; V_2(\mu) = \frac{\sin\mu - \mu\cos\mu}{\mu^2}. \end{gathered} \tag{2}$$

Equation (1) is transcendental and cannot be solved in analytic form. Its first six roots have been tabulated in [3] for 40 different values of the Biot number. Grigull presents more accurate tables for 14 Biot numbers in [4, 5].

Nonequilibrium heat-conduction calculations are usually made with a computer. It is not good practice to introduce roots into the computer in tabular form, because tables load the memory, and selection and interpolation of the required values involves considerable time. It is much more convenient to use Chebyshev approximations [6], particularly because determination of their coefficients in terms of tabulated values can easily be done by means of a program [7].

For proper choice of the Chebyshev polynomial argument we first find asymptotic expressions for μ_n ($n = 1, 2, \ldots, \infty$). For this purpose, we examine two limiting cases; $\mu \to 0$ and $\mathrm{Bi} \to \infty$.

Using the series derived in [1, 2] for $W_{\Gamma}(\mu)$ and $V_{\Gamma}(\mu)$, it can be shown that

$$\frac{W_{\Gamma}(\mu)}{V_{\Gamma}(\mu)} = \frac{\Gamma+1}{\mu} - \frac{\mu}{\Gamma+3} - \frac{\mu^3}{(\Gamma+3)^2(\Gamma+5)} \cdot \frac{W_{\Gamma+6}(\mu)}{W_{\Gamma+2}(\mu)}. \tag{3}$$

For $\mu \to 0$ we can neglect the last term on the right side of Eq. (3). Substituting the result in Eq. (1), we obtain the following asymptotic expression for the first root:

$$\mu_1^2 = \sqrt{(\Gamma+1)(\Gamma+3)}\sqrt{\frac{\mathrm{Bi}}{\Gamma+3+\mathrm{Bi}}}. \tag{4}$$

For $\mathrm{Bi} \to \infty$ the roots in Eq. (1) are given by

$$\mu_n^{(1)} = \left(n - \frac{1}{2} + \frac{\Gamma}{4}\right)\pi. \tag{5}$$

Eq. (5) is exact for $\Gamma = 0$ and 2. It is asymptotic for $\Gamma = 1$.

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TABLE 1. Coefficients $\alpha_{n,k}$ for a Plate

k	$n=1$	$n=2$	$n=3$	$n=4$
0	$-0{,}74624414$	$-0{,}74750704$	$-0{,}76353794$	$-0{,}77003115$
1	$+0{,}79756431$	$+0{,}80698354$	$+0{,}81697734$	$+0{,}82063950$
2	$-0{,}37660689\cdot10^{-1}$	$-0{,}40242511\cdot10^{-1}$	$-0{,}24299633\cdot10^{-1}$	$-0{,}17369540\cdot10^{-1}$
3	$-0{,}12582065\cdot10^{-1}$	$-0{,}22405683\cdot10^{-1}$	$-0{,}33638841\cdot10^{-1}$	$-0{,}37943919\cdot10^{-1}$
4	$-0{,}16754416\cdot10^{-2}$	$+0{,}24330272\cdot10^{-2}$	$+0{,}26842460\cdot10^{-2}$	$+0{,}22495208\cdot10^{-2}$
5	$+0{,}40976275\cdot10^{-3}$	$+0{,}85835975\cdot10^{-3}$	$+0{,}22070060\cdot10^{-2}$	$+0{,}29382952\cdot10^{-2}$
6	$+0{,}19234612\cdot10^{-3}$	$-0{,}80406665\cdot10^{-4}$	$-0{,}26697125\cdot10^{-3}$	$-0{,}27608311\cdot10^{-3}$
7	$+0{,}10221381\cdot10^{-4}$	$-0{,}40531580\cdot10^{-4}$	$-0{,}15802503\cdot10^{-3}$	$-0{,}25726076\cdot10^{-3}$
8	$-0{,}11479656\cdot10^{-4}$	$-0{,}16632984\cdot10^{-5}$	$+0{,}23983797\cdot10^{-4}$	$+0{,}32248470\cdot10^{-4}$
9	$-0{,}32257521\cdot10^{-5}$	$+0{,}26601483\cdot10^{-5}$	$+0{,}11554453\cdot10^{-4}$	$+0{,}23556334\cdot10^{-4}$
10	$+0{,}37543941\cdot10^{-7}$	$+0{,}46090281\cdot10^{-6}$	$-0{,}18110731\cdot10^{-5}$	$-0{,}33192773\cdot10^{-5}$
11	$+0{,}86850195\cdot10^{-6}$	$-0{,}16436388\cdot10^{-6}$	$-0{,}85520150\cdot10^{-6}$	$-0{,}20338047\cdot10^{-5}$
Max.abs. error	$2{,}9\cdot10^{-6}$	$1{,}7\cdot10^{-7}$	$4{,}1\cdot10^{-7}$	$1{,}4\cdot10^{-6}$

k	$n=5$	$n=6$	$n=7$	$n=8$
0	$-0{,}77354935$	$-0{,}77575640$	$-0{,}77727025$	$-0{,}77837319$
1	$+0{,}82253134$	$+0{,}82368503$	$+0{,}82446165$	$+0{,}82501994$
2	$-0{,}13509255\cdot10^{-1}$	$-0{,}11051137\cdot10^{-1}$	$-0{,}93492308\cdot10^{-2}$	$-0{,}81012953\cdot10^{-2}$
3	$-0{,}40203357\cdot10^{-1}$	$-0{,}41592358\cdot10^{-1}$	$-0{,}42531814\cdot10^{-1}$	$-0{,}43209291\cdot10^{-1}$
4	$+0{,}18848019\cdot10^{-2}$	$+0{,}16099018\cdot10^{-2}$	$+0{,}14010153\cdot10^{-2}$	$+0{,}12384170\cdot10^{-2}$
5	$+0{,}33638764\cdot10^{-2}$	$+0{,}36386559\cdot10^{-2}$	$+0{,}38298638\cdot10^{-2}$	$+0{,}39703721\cdot10^{-2}$
6	$-0{,}25370052\cdot10^{-3}$	$-0{,}22839634\cdot10^{-3}$	$-0{,}20559242\cdot10^{-3}$	$-0{,}18608654\cdot10^{-3}$
7	$-0{,}32342800\cdot10^{-3}$	$-0{,}36900397\cdot10^{-3}$	$-0{,}40193469\cdot10^{-3}$	$-0{,}42673242\cdot10^{-3}$
8	$+0{,}32954019\cdot10^{-4}$	$+0{,}31469535\cdot10^{-4}$	$+0{,}29376780\cdot10^{-4}$	$+0{,}27309593\cdot10^{-4}$
9	$+0{,}32868614\cdot10^{-4}$	$+0{,}39896738\cdot10^{-4}$	$+0{,}45184278\cdot10^{-4}$	$+0{,}49316746\cdot10^{-4}$
10	$-0{,}40973245\cdot10^{-5}$	$-0{,}42977335\cdot10^{-5}$	$-0{,}42266910\cdot10^{-5}$	$-0{,}41008461\cdot10^{-5}$
11	$-0{,}31283125\cdot10^{-5}$	$-0{,}40505110\cdot10^{-5}$	$-0{,}47769281\cdot10^{-5}$	$-0{,}53830735\cdot10^{-5}$
Max.abs. error	$2\cdot10^{-6}$	$2{,}2\cdot10^{-6}$	$2{,}5\cdot10^{-6}$	$2{,}5\cdot10^{-6}$

k	$n=9$	$n=10$
0	$-0{,}77921253$	$-0{,}77987278$
1	$+0{,}82544055$	$+0{,}82576888$
2	$-0{,}71471631\cdot10^{-2}$	$-0{,}63940161\cdot10^{-2}$
3	$-0{,}43720850\cdot10^{-1}$	$-0{,}44120573\cdot10^{-1}$
4	$+0{,}11088308\cdot10^{-2}$	$+0{,}10032999\cdot10^{-2}$
5	$+0{,}40778259\cdot10^{-2}$	$+0{,}41625995\cdot10^{-2}$
6	$-0{,}16953461\cdot10^{-3}$	$-0{,}15552209\cdot10^{-3}$
7	$-0{,}44606678\cdot10^{-3}$	$-0{,}46149911\cdot10^{-3}$
8	$+0{,}25358938\cdot10^{-4}$	$+0{,}23708809\cdot10^{-4}$
9	$+0{,}52585324\cdot10^{-4}$	$+0{,}55194934\cdot10^{-4}$
10	$-0{,}39704609\cdot10^{-5}$	$-0{,}38064463\cdot10^{-5}$
11	$-0{,}58727891\cdot10^{-5}$	$-0{,}63863844\cdot10^{-5}$
Max.abs. error	$2{,}4\cdot10^{-6}$	$2{,}4\cdot10^{-6}$

The correction to the eigenvalue $\mu_n^{(1)}$ is given by

$$\mu_n = \mu_n^{(1)} - \frac{\varepsilon}{\mathrm{Bi}}, \tag{6}$$

where ε is some number.

Using Taylor's series, we can write

$$W_\Gamma\left(\mu_n^{(1)} - \frac{\varepsilon}{\mathrm{Bi}}\right) = V_\Gamma(\mu_n^{(1)})\left\{\frac{\varepsilon}{\mathrm{Bi}} + \frac{\Gamma}{2\mu_n^{(1)}}\left(\frac{\varepsilon}{\mathrm{Bi}}\right)^2 + \ldots\right\}, \tag{7}$$

$$V_\Gamma\left(\mu_n^{(1)} - \frac{\varepsilon}{\mathrm{Bi}}\right) = V_\Gamma(\mu_n^{(1)})\left\{1 + \frac{\Gamma}{\mu_n^{(1)}}\frac{\varepsilon}{\mathrm{Bi}} + \ldots\right\}. \tag{8}$$

TABLE 2. Coefficients $\alpha_{n,k}$ for a Cylinder

k	$n=1$	$n=2$	$n=3$	$n=4$
0	$-0,10539837\cdot 10$	$-0,78416870$	$-0,78305778$	$-0,78322982$
1	$+0,12305030\cdot 10$	$+0,87675207$	$+0,85674086$	$+0,84848454$
2	$-0,98448116\cdot 10^{-1}$	$-0,41852796\cdot 10^{-1}$	$-0,24625005\cdot 10^{-1}$	$-0,17486527\cdot 10^{-1}$
3	$-0,30111669\cdot 10^{-1}$	$-0,34330628\cdot 10^{-1}$	$-0,40527444\cdot 10^{-1}$	$-0,42766895\cdot 10^{-1}$
4	$-0,18850414\cdot 10^{-2}$	$+0,44811004\cdot 10^{-2}$	$+0,33280149\cdot 10^{-2}$	$+0,25566518\cdot 10^{-2}$
5	$+0,21094276\cdot 10^{-2}$	$+0,18474684\cdot 10^{-2}$	$+0,30956798\cdot 10^{-2}$	$+0,36321480\cdot 10^{-2}$
6	$+0,60997885\cdot 10^{-3}$	$-0,36514209\cdot 10^{-3}$	$-0,40921501\cdot 10^{-3}$	$-0,35444401\cdot 10^{-3}$
7	$-0,81636040\cdot 10^{-4}$	$-0,99093565\cdot 10^{-4}$	$-0,25945124\cdot 10^{-3}$	$-0,35025733\cdot 10^{-3}$
8	$-0,81922538\cdot 10^{-4}$	$+0,23946398\cdot 10^{-4}$	$+0,46789646\cdot 10^{-4}$	$+0,46335975\cdot 10^{-4}$
9	$-0,10153191\cdot 10^{-4}$	$+0,34377008\cdot 10^{-5}$	$+0,21869011\cdot 10^{-4}$	$+0,34861877\cdot 10^{-4}$
10	$+0,55968412\cdot 10^{-5}$	$+0,13233075\cdot 10^{-5}$	$-0,35739358\cdot 10^{-5}$	$-0,60327438\cdot 10^{-5}$
11	$+0,37323916\cdot 10^{-5}$	$-0,74979179\cdot 10^{-5}$	$-0,87665113\cdot 10^{-6}$	$-0,33941905\cdot 10^{-5}$
Max.abs. error	$4,8\cdot 10^{-6}$	$9,6\cdot 10^{-6}$	$3,9\cdot 10^{-6}$	$4,1\cdot 10^{-6}$

k	$n=5$	$n=6$	$n=7$	$n=8$
0	$-0,78349528$	$-0,78372850$	$-0,78391913$	$-0,78407414$
1	$+0,84396204$	$+0,84110614$	$+0,83913806$	$+0,83769926$
2	$-0,13562604\cdot 10^{-1}$	$-0,11080178\cdot 10^{-1}$	$-0,93667328\cdot 10^{-2}$	$-0,81126870\cdot 10^{-2}$
3	$-0,43911369\cdot 10^{-1}$	$-0,44603211\cdot 10^{-1}$	$-0,45066021\cdot 10^{-1}$	$-0,45397302\cdot 10^{-1}$
4	$+0,20640987\cdot 10^{-2}$	$+0,17268415\cdot 10^{-2}$	$+0,14830945\cdot 10^{-2}$	$+0,12991171\cdot 10^{-2}$
5	$+0,39229438\cdot 10^{-2}$	$+0,41058253\cdot 10^{-2}$	$+0,42300507\cdot 10^{-2}$	$+0,43202663\cdot 10^{-2}$
6	$-0,30111930\cdot 10^{-3}$	$-0,26053146\cdot 10^{-3}$	$-0,22862065\cdot 10^{-3}$	$-0,20325988\cdot 10^{-3}$
7	$-0,40433800\cdot 10^{-3}$	$-0,43961749\cdot 10^{-3}$	$-0,46387749\cdot 10^{-3}$	$-0,48163204\cdot 10^{-3}$
8	$+0,42577841\cdot 10^{-4}$	$+0,39007369\cdot 10^{-4}$	$+0,35092438\cdot 10^{-4}$	$+0,31790768\cdot 10^{-4}$
9	$+0,42822401\cdot 10^{-4}$	$+0,49116235\cdot 10^{-4}$	$+0,53985772\cdot 10^{-4}$	$+0,57106270\cdot 10^{-4}$
10	$-0,45256601\cdot 10^{-5}$	$-0,49142254\cdot 10^{-5}$	$-0,47417124\cdot 10^{-5}$	$-0,48238580\cdot 10^{-5}$
11	$-0,58712903\cdot 10^{-5}$	$-0,60474703\cdot 10^{-5}$	$-0,67373912\cdot 10^{-5}$	$-0,69676617\cdot 10^{-5}$
Max.abs. error	$5,2\cdot 10^{-6}$	$2,3\cdot 10^{-6}$	$2,5\cdot 10^{-6}$	$2,7\cdot 10^{-6}$

k	$n=9$	$n=10$
0	$-0,78420150$	$-0,78430728$
1	$+0,83660153$	$+0,83573595$
2	$-0,71549217\cdot 10^{-2}$	$-0,63995175\cdot 10^{-2}$
3	$-0,45646052\cdot 10^{-1}$	$-0,45838924\cdot 10^{-1}$
4	$+0,11555363\cdot 10^{-2}$	$+0,10401151\cdot 10^{-2}$
5	$+0,43886872\cdot 10^{-2}$	$+0,44420132\cdot 10^{-2}$
6	$-0,18287885\cdot 10^{-3}$	$-0,16578678\cdot 10^{-3}$
7	$-0,49536542\cdot 10^{-3}$	$-0,50639234\cdot 10^{-3}$
8	$+0,29237053\cdot 10^{-4}$	$+0,26618217\cdot 10^{-4}$
9	$+0,59540791\cdot 10^{-4}$	$+0,61917584\cdot 10^{-4}$
10	$-0,51505776\cdot 10^{-5}$	$-0,49579830\cdot 10^{-5}$
11	$-0,71174436\cdot 10^{-5}$	$-0,75021816\cdot 10^{-6}$
Max.abs. error	$3\cdot 10^{-6}$	$3\cdot 10^{-6}$

Substituting (7) and (8) into Eq. (1) and neglecting terms containing $(1/\mathrm{Bi})^2$ and higher powers, of $1/\mathrm{Bi}$, we obtain

$$\mu_n = \left(n - \frac{1}{2} + \frac{\Gamma}{4}\right)\pi\left(1 - \frac{1}{\mathrm{Bi}}\right). \tag{9}$$

For $\Gamma = 0$, after squaring and neglecting terms containing $1/\mathrm{Bi}^2$, we obtain from Eq. (9) the expression given in [8], p. 151.

In the present work, the minimax approximation for the n-th root of Eq. (1) is constructed in the form of a series of Chebyshev polynomials

$$\mu_1 = \sum_{k=0}^{11} \alpha_{1,k} T_k(2x - 1), \tag{10}$$

TABLE 3. Coefficients $\alpha_{n,k}$ for a Sphere

k	$n=1$	$n=2$	$n=3$	$n=4$
0	$-0{,}14022951\cdot 10$	$-0{,}85576886$	$-0{,}82761458$	$-0{,}81563785$
1	$+0{,}16145633\cdot 10$	$+0{,}93706708$	$+0{,}89305064$	$+0{,}87455098$
2	$-0{,}16916148$	$-0{,}44863540\cdot 10^{-1}$	$-0{,}25655128\cdot 10^{-1}$	$-0{,}18004166\cdot 10^{-1}$
3	$-0{,}48144595\cdot 10^{-1}$	$-0{,}45073769\cdot 10^{-1}$	$-0{,}46969392\cdot 10^{-1}$	$-0{,}47360943\cdot 10^{-1}$
4	$-0{,}14194991\cdot 10^{-3}$	$+0{,}64328810\cdot 10^{-2}$	$+0{,}40140483\cdot 10^{-2}$	$+0{,}29031514\cdot 10^{-2}$
5	$+0{,}47345985\cdot 10^{-2}$	$+0{,}30817344\cdot 10^{-2}$	$+0{,}40290063\cdot 10^{-2}$	$+0{,}43435881\cdot 10^{-2}$
6	$+0{,}98167703\cdot 10^{-3}$	$-0{,}75648704\cdot 10^{-3}$	$-0{,}57541838\cdot 10^{-3}$	$-0{,}44194837\cdot 10^{-3}$
7	$-0{,}36675442\cdot 10^{-3}$	$-0{,}19684208\cdot 10^{-3}$	$-0{,}38058366\cdot 10^{-3}$	$-0{,}45378714\cdot 10^{-3}$
8	$-0{,}20361572\cdot 10^{-3}$	$+0{,}73507413\cdot 10^{-4}$	$+0{,}77912147\cdot 10^{-4}$	$+0{,}63635496\cdot 10^{-4}$
9	$+0{,}17347920\cdot 10^{-3}$	$+0{,}94432034\cdot 10^{-5}$	$+0{,}36152734\cdot 0^{-4}$	$+0{,}52013361\cdot 10^{-4}$
10	$+0{,}29619172\cdot 10^{-4}$	$-0{,}78587181\cdot 10^{-5}$	$-0{,}81947946\cdot 10^{-5}$	$-0{,}10453877\cdot 10^{-4}$
11	$+0{,}80290192\cdot 10^{-5}$	$+0{,}29509828\cdot 10^{-6}$	$-0{,}27387432\cdot 10^{-5}$	$-0{,}73544506\cdot 10^{-5}$
Max. abs. error	$1{,}3\cdot 10^{-6}$	$6{,}2\cdot 10^{-6}$	$5{,}2\cdot 10^{-6}$	$7{,}6\cdot 10^{-4}$

k	$n=5$	$n=6$	$n=7$	$n=8$
0	$-0{,}80897128$	$-0{,}80471916$	$-0{,}80176838$	$-0{,}79960007$
1	$+0{,}86431175$	$+0{,}85780055$	$+0{,}85329235$	$+0{,}84998542$
2	$-0{,}13876583\cdot 10^{-1}$	$-0{,}11289290\cdot 10^{-1}$	$-0{,}95160497\cdot 10^{-2}$	$-0{,}82247517\cdot 10^{-2}$
3	$-0{,}47480128\cdot 10^{-1}$	$-0{,}47521042\cdot 10^{-1}$	$-0{,}47533494\cdot 10^{-1}$	$-0{,}47535180\cdot 10^{-1}$
4	$+0{,}22739112\cdot 10^{-2}$	$+0{,}18670662\cdot 10^{-2}$	$+0{,}15830749\cdot 10^{-2}$	$+0{,}13736667\cdot 10^{-2}$
5	$+0{,}44930226\cdot 10^{-2}$	$+0{,}45795110\cdot 10^{-2}$	$+0{,}46352024\cdot 10^{-2}$	$+0{,}46737378\cdot 10^{-2}$
6	$-0{,}35846841\cdot 10^{-3}$	$-0{,}29978637\cdot 10^{-3}$	$-0{,}25768985\cdot 10^{-3}$	$-0{,}22516958\cdot 10^{-3}$
7	$-0{,}49125005\cdot 10^{-3}$	$-0{,}51373736\cdot 10^{-3}$	$-0{,}52861562\cdot 10^{-3}$	$-0{,}53826803\cdot 10^{-3}$
8	$+0{,}54509015\cdot 10^{-4}$	$+0{,}46714398\cdot 10^{-4}$	$+0{,}40886050\cdot 10^{-4}$	$+0{,}36502649\cdot 10^{-4}$
9	$+0{,}57599361\cdot 10^{-4}$	$+0{,}62214094\cdot 10^{-4}$	$+0{,}65468164\cdot 10^{-4}$	$+0{,}66627239\cdot 10^{-4}$
10	$-0{,}78877783\cdot 10^{-5}$	$-0{,}70629612\cdot 10^{-5}$	$-0{,}53734547\cdot 10^{-5}$	$-0{,}49601076\cdot 10^{-5}$
11	$-0{,}62923791\cdot 10^{-5}$	$-0{,}71512040\cdot 10^{-5}$	$-0{,}80850004\cdot 10^{-6}$	$-0{,}89031382\cdot 10^{-5}$
Max. abs. error	$7{,}3\cdot 10^{-6}$	$5{,}2\cdot 10^{-6}$	$5{,}9\cdot 10^{-6}$	$3{,}4\cdot 10^{-6}$

k	$n=9$	$n=10$
0	$-0{,}79793957$	$-0{,}79662694$
1	$+0{,}84745530$	$+0{,}84545707$
2	$-0{,}72419405\cdot 10^{-2}$	$-0{,}64690723\cdot 10^{-2}$
3	$-0{,}47531117\cdot 10^{-1}$	$-0{,}47525145\cdot 10^{-1}$
4	$+0{,}12137467\cdot 10^{-2}$	$+0{,}10870648\cdot 10^{-2}$
5	$+0{,}47021039\cdot 10^{-2}$	$+0{,}47237517\cdot 10^{-2}$
6	$-0{,}19999685\cdot 10^{-3}$	$-0{,}17993908\cdot 10^{-3}$
7	$-0{,}54626257\cdot 10^{-3}$	$-0{,}55243236\cdot 10^{-3}$
8	$+0{,}33099961\cdot 10^{-4}$	$+0{,}30326744\cdot 10^{-4}$
9	$+0{,}68271314\cdot 10^{-4}$	$+0{,}69659246\cdot 10^{-4}$
10	$-0{,}40085142\cdot 10^{-5}$	$-0{,}35457342\cdot 10^{-5}$
11	$-0{,}89633395\cdot 10^{-5}$	$-0{,}90831890\cdot 10^{-5}$
Max. abs. error	$3{,}4\cdot 10^{-6}$	$3{,}4\cdot 10^{-6}$

where

$$x=\sqrt{\frac{\mathrm{Bi}}{\Gamma+3+\mathrm{Bi}}} \tag{11}$$

and

$$\mu_n=\left(n-\frac{1}{2}+\frac{\Gamma}{4}\right)\pi+\sum_{k=0}^{11}\alpha_{n,k}T_k(2x-1), \tag{12}$$

where

$$x=\frac{\mathrm{Bi}}{\left(n-\frac{1}{2}+\frac{\Gamma}{4}\right)\pi+\mathrm{Bi}}. \tag{13}$$

The series (10) and (12) are selected by comparing the first two terms with the asymptotic expressions (4) and (9).

Tables 1-3 list the coefficients $\alpha_{n,k}$ for the first 10 roots for the plate cylinder, and sphere, respectively, as well as the maximum absolute approximation error. The latter is determined by comparing the roots of Eq. (1) with those obtained from (10) and (12) for $x = 0$ to 1 at intervals pf 0.01. The interval is 0.005 for the first four roots in the spherical case. For verification, the roots of Eq. (1) were calculated to 10^{-8}. The results were obtained by means of the program described in [7].

The roots at the approximation points were calculated to 10^{-7}. For the plate and cylinder, 81 approximation points were used for the first two roots and 61 for the others. For the sphere, 81 points were used for all roots.

The series (10) and (12) agree with the tabulated values of roots given in [4] to six significant figures.

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